Predicate Logic

Zhaoguo Wang

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Stanford CS103 Mathematical Logic
教材《数理逻辑与集合论》第4章、第5章

Predicate Logic（谓词逻辑）

1.1 Propositional Logic

Step 0. Proof.

Step 1. Convert program into mathematical formula.

Step 2. Ask the computer to solve the formula.

1.2 First Order Logic

Step 1. Convert it into first logic formula.

Step 2. Ask the computer to solve the formula.

1.3 Interactive Proof

Step 1. Axiom system

Step 2. Ask the computer to check the invariants

A Quick Recap – Propositional Logic

- Connectives
- Truth table
- SAT
- DPLL
- …

Outline – This Lecture

- Basic Concept
- Deduction
- SMT Solver

Outline – This Lecture

- Basic Concept
- Deduction
- SMT Solver

Propositional Logic

This is complicated.

```
void f(bool a, bool b)
{
    unsigned x, y;
    if (a)
        x = 1;
    else
        x = 0;
    if (b)
        y = 1;
    else
        y = 0;
    assert(x + y > 0);
}
```

$$formula\{X1 = [00...01]\}\wedge$$
$$formula\{X2 = [00...00]\}\wedge$$
$$formula\{Y4 = [00...01]\}\wedge$$
$$formula\{Y5 = [00...00]\}\wedge$$
$$\wedge_{i=0,\ldots,31}\,(X3_i \leftrightarrow (A \wedge X1_i) \vee (\neg A \wedge X2_i))\wedge$$
$$\wedge_{i=0,\ldots,31}\,(Y6_i \leftrightarrow (B \wedge Y4_i) \vee (\neg B \wedge Y5_i)) \rightarrow$$
$$formula\{X3 + Y6 > 0\}$$

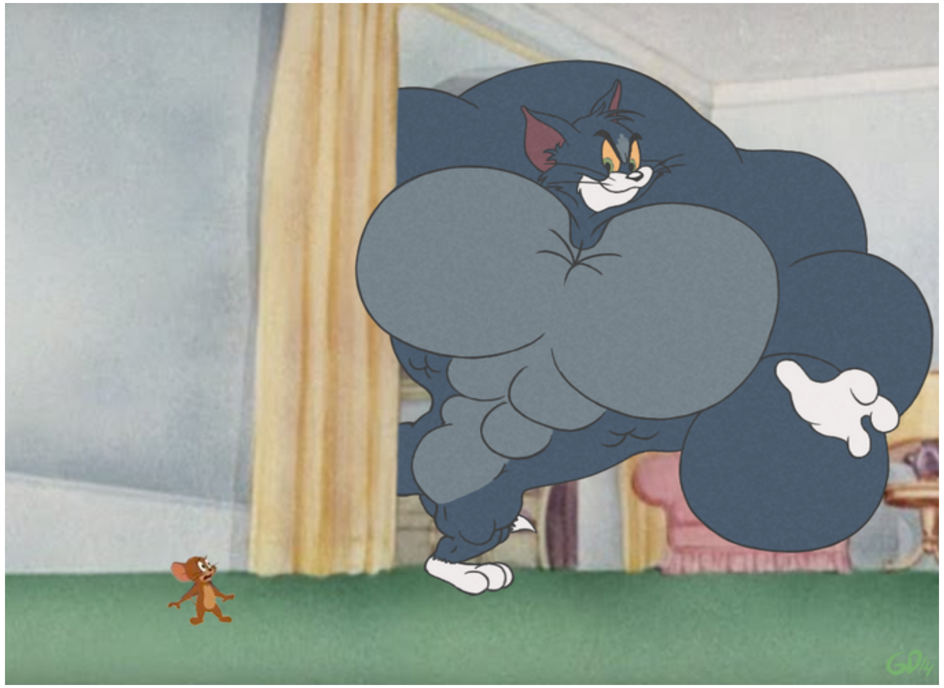
How to simplify it?

Predicate（谓词）

DEFINITION

Predicates describe properties of individuals（个体词）.

The range of individuals is domain of discourse（论域）.



individuals

$$STRONGER(Tom, Jerry)$$

We use upper-case words to represent predicates.

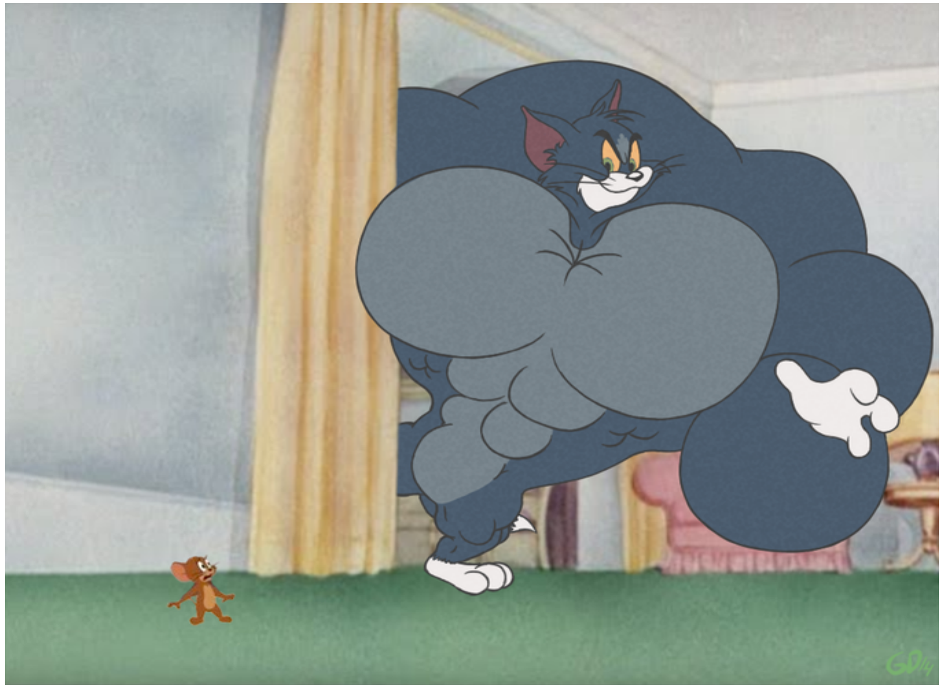
Predicate（谓词）

DEFINITION

Predicates describe properties of individuals（个体词）.

The range of individuals is domain of discours

Tom and Jerry are individual constants.（个体常项）



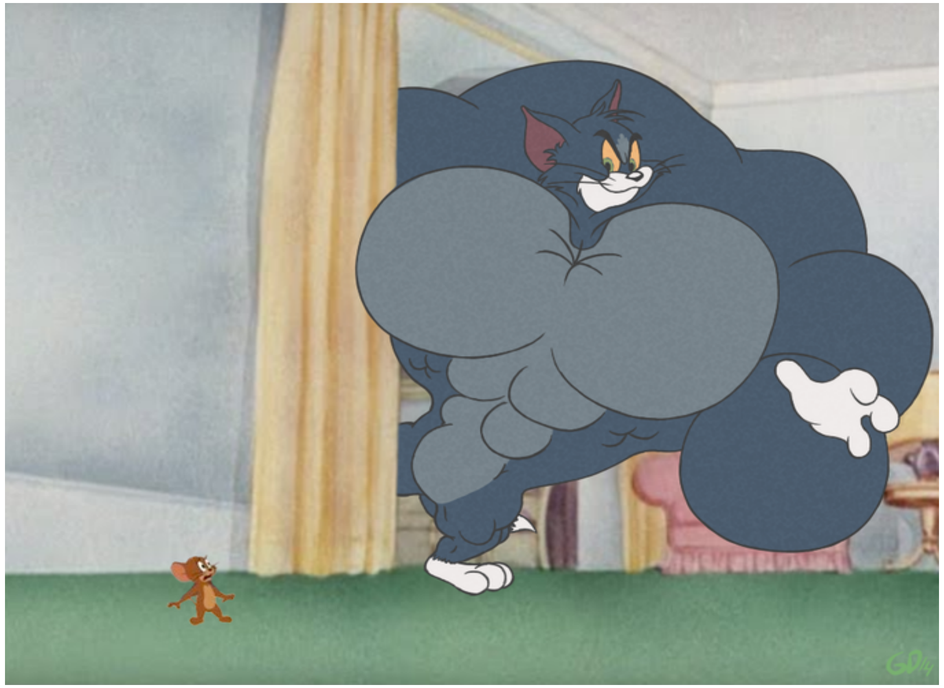
$$STRONGER(Tom, Jerry)$$

STRONGER is a predicate constant.（谓词常项）

Predicate（谓词）

NOTE

Each predicate maps individuals to T or F.



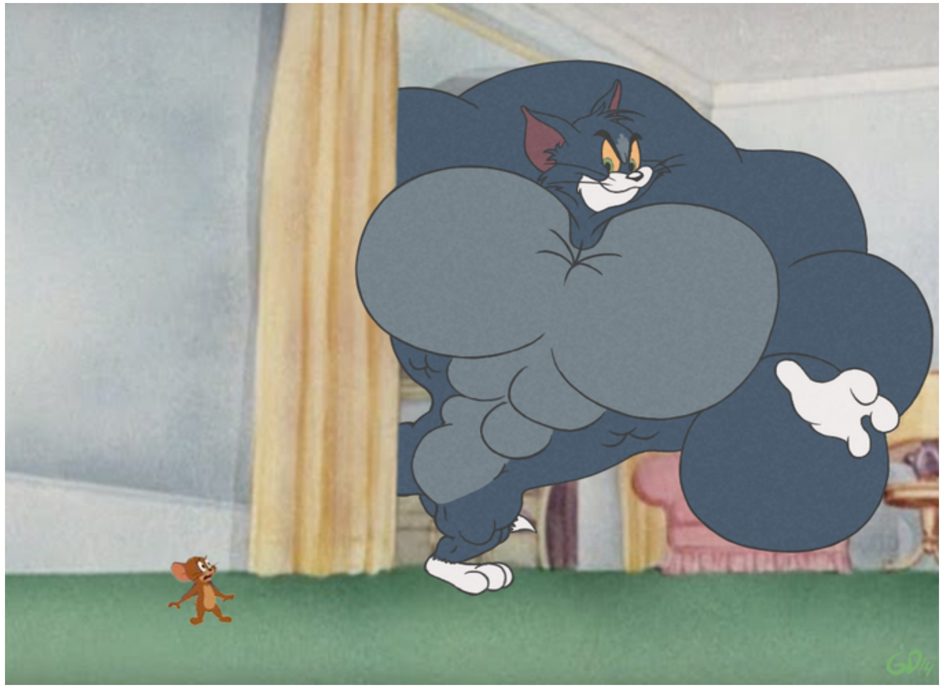
STRONGER maps Tom and Jerry to T or F.

$$STRONGER(Tom, Jerry)$$

Predicate（谓词）

NOTE

Each predicate has an associated arity, a natural number indicating how many arguments it takes.



$$STRONGER(Tom, Jerry)$$

STRONGER takes two arguments.

Predicate（谓词）

NOTE

Each predicate has an associated arity, a natural number indicating how many arguments it takes.

$$P(x_1, ..., x_n)$$

n-ary predicate
（n元谓词）

$$7 = 5$$

Equality is a special predicate of arity 2.

Predicate（谓词）

$$P(x_1, ..., x_n)$$

It is not a proposition. Because P is a predicate variable（谓词变项） and xs are individual variables（个体变项）. It is a proposition only when they are all constants.

Predicate（谓词）

$$
\begin{array}{ll}
formula\{X1 = [00...01]\}\wedge & x1 = 1\wedge \\
formula\{X2 = [00...00]\}\wedge & x2 = 0\wedge \\
formula\{Y4 = [00...01]\}\wedge & y4 = 1\wedge \\
formula\{Y5 = [00...00]\}\wedge \quad \Rightarrow & y5 = 0\wedge \\
\wedge_{i=0,\dots,31} (X3_i \leftrightarrow (A \wedge X1_i) \vee (\neg A \wedge X2_i))\wedge & \wedge_{i=0,\dots,31} (x3_i = (a \wedge x1_i) \vee (\neg a \wedge x2_i))\wedge \\
\wedge_{i=0,\dots,31} (Y6_i \leftrightarrow (B \wedge Y4_i) \vee (\neg B \wedge Y5_i)) \rightarrow & \wedge_{i=0,\dots,31} (y6_i = (b \wedge y4_i) \vee (\neg b \wedge y5_i)) \rightarrow \\
formula\{X3 + Y6 > 0\} & formula\{x3 + y6 > 0\}
\end{array}
$$

After introducing predicates

Predicate（谓词）

$$
\begin{aligned}
&formula\{X1 = [00...01]\}\wedge \\
&formula\{X2 = [00...00]\}\wedge \\
&formula\{Y4 = [00...01]\}\wedge \\
&formula\{Y5 = [00...00]\}\wedge \\
&\textstyle\bigwedge_{i=0,\ldots,31} (X3_i \leftrightarrow (A \wedge X1_i) \vee (\neg A \wedge X2_i))\wedge \\
&\textstyle\bigwedge_{i=0,\ldots,31} (Y6_i \leftrightarrow (B \wedge Y4_i) \vee (\neg B \wedge Y5_i)) \rightarrow \\
&formula\{X3 + Y6 > 0\}
\end{aligned}
$$

$$
\begin{aligned}
&x1 = 1\wedge \\
&x2 = 0\wedge \\
&y4 = 1\wedge \\
&y5 = 0\wedge \\
&\textstyle\bigwedge_{i=0,\ldots,31} (x3_i = (a \wedge x1_i) \vee (\neg a \wedge x2_i))\wedge \\
&\textstyle\bigwedge_{i=0,\ldots,31} (y6_i = (b \wedge y4_i) \vee (\neg b \wedge y5_i)) \rightarrow \\
&formula\{x3 + y6 > 0\}
\end{aligned}
$$

Can we go further?

Function（函数）

DEFINITION

A function maps individuals to an individual instead of T or F.

Functions are represented by lower-case words.

Function with one argument

$$bestfriend(SpongeBob) = PatrickStar$$

predicate

Function（函数）

DEFINITION

A function maps individuals to an individual instead of T or F.

Functions are represented by lower-case words.

Not a proposition

$$\boxed{bestfriend(SpongeBob)} = PatrickStar$$

Function（函数）

DEFINITION

A function maps individuals to an individual instead of T or F.

Functions are represented by lower-case words.

a proposition

$$\boxed{bestfriend(SpongeBob) = PatrickStar}$$

Function（函数）

When working in predicate logic, be careful to keep individuals (actual things) and propositions (true or false) separate.

$$bestfriend(SpongeBob) \overset{⚠}{\rightarrow} PatrickStar$$

Don't use connectives to associate individuals.

$$\overset{⚠}{not}(T) = F$$

Functions cannot operate on propositions.

The Type-Checking Table

	arguments	results
connectives	propositions	a proposition
predicates	individuals	a proposition
functions	individuals	an individual

Examples

Predicate logic is an expansion of propositional logic.

Propositional logic

- Equality: no
- Predicates: P, Q, R, …
- Functions: no

Propositional variables can be treated as nullary predicates.

Number theory

- Equality: yes
- Predicates: >, <, …
- Functions: +, -, …

Function（函数）

$x1 = 1 \wedge$

$x2 = 0 \wedge$

$y4 = 1 \wedge$

$y5 = 0 \wedge$

$\wedge_{i=0,\ldots,31} (x3_i = (a \wedge x1_i) \vee (\neg a \wedge x2_i)) \wedge$

$\wedge_{i=0,\ldots,31} (y6_i = (b \wedge y4_i) \vee (\neg b \wedge y5_i)) \rightarrow$

$formula\{x3 + y6 > 0\}$

Ite: a?x1:x2

$x1 = 1 \wedge$

$x2 = 0 \wedge$

$y4 = 1 \wedge$

$y5 = 0 \wedge$

$x3 = ite(a, x1, x2) \wedge$

$y6 = ite(b, y4, y5) \rightarrow$

$sum(x3, y6) > 0$

Function（函数）

$$
\begin{aligned}
& x1 = 1 \wedge \\
& x2 = 0 \wedge \\
& y4 = 1 \wedge \\
& y5 = 0 \wedge \\
& \wedge_{i=0,\ldots,31} (x3_i = (a \wedge x1_i) \vee (\neg a \wedge x2_i)) \wedge \\
& \wedge_{i=0,\ldots,31} (y6_i = (b \wedge y4_i) \vee (\neg b \wedge y5_i)) \rightarrow \\
& formula\{x3 + y6 > 0\}
\end{aligned}
$$

$$
\begin{aligned}
& x1 = 1 \wedge \\
& x2 = 0 \wedge \\
& y4 = 1 \wedge \\
& y5 = 0 \wedge \\
& x3 = ite(a, x1, x2) \wedge \\
& y6 = ite(b, y4, y5) \rightarrow \\
& sum(x3, y6) > 0
\end{aligned}
$$

We reserve logical connectives.

Function（函数）

$$
\begin{aligned}
&x1 = 1 \wedge \\
&x2 = 0 \wedge \\
&y4 = 1 \wedge \\
&y5 = 0 \wedge \\
&x3 = ite(a, x1, x2) \wedge \\
&y6 = ite(b, y4, y5) \rightarrow \\
&sum(x3, y6) > 0
\end{aligned}
$$

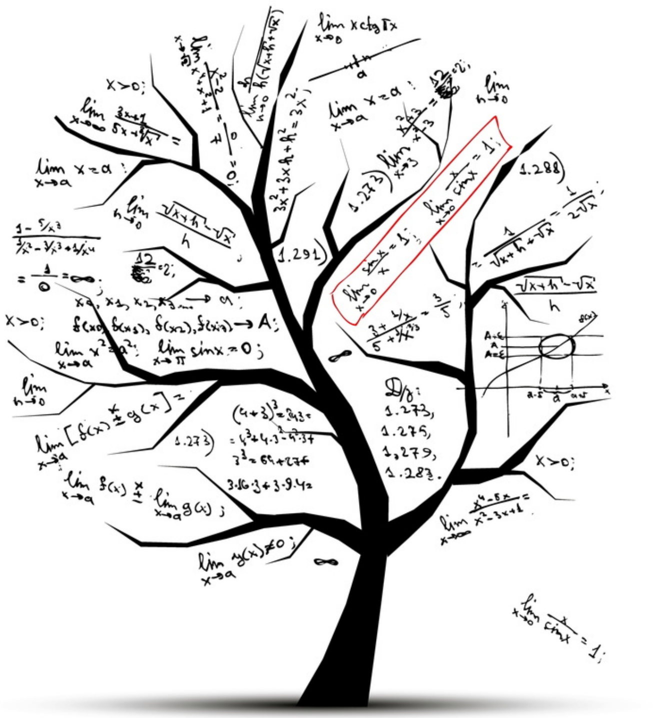
There are some individual variables. It is not a real proposition. But we cannot replace them with constants.

How to express “for all integers”?

Quantifier（量词）

DEFINITION

A quantifier turns a formula about individuals having some property into a formula about the number (quantity) of individuals having the property.

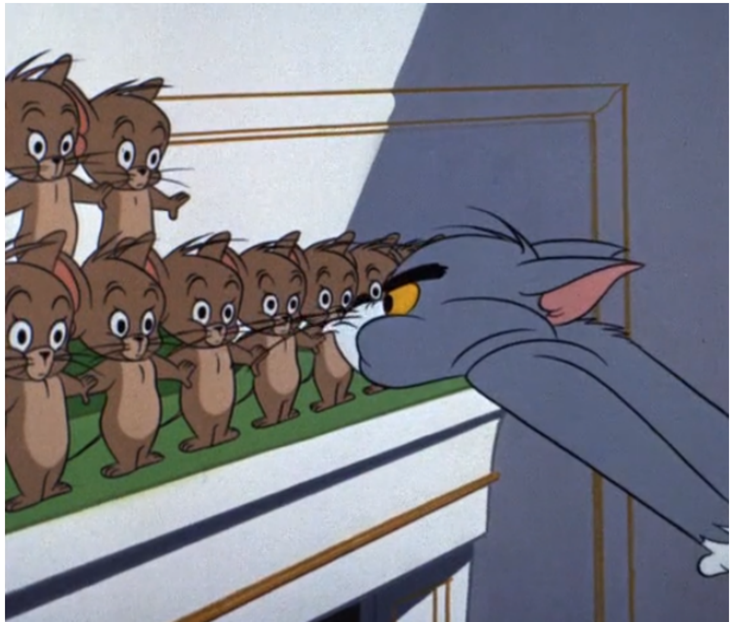


We have met quantifiers in mathematics.

The Universal Quantifier（全称量词）

DEFINITION

The universal quantifier expresses that a proposition can be satisfied by every member of the domain of discourse, which is interpreted as "given any" or "for all".



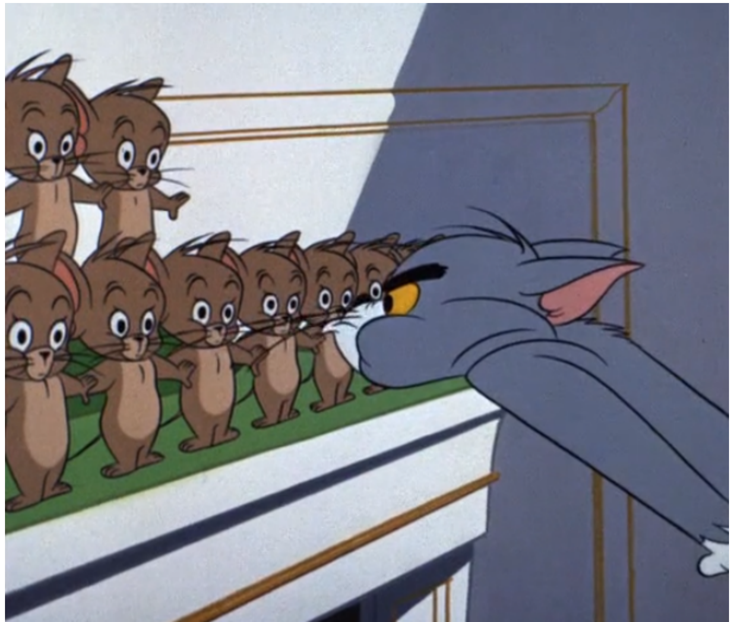
All toys look like Jerry.

$$(\forall x)(LIKEJERRY(x))$$

The Universal Quantifier（全称量词）

DEFINITION

The universal quantifier expresses that a proposition can be satisfied by every member of the domain of discourse, which is interpreted as "given any" or "for all".



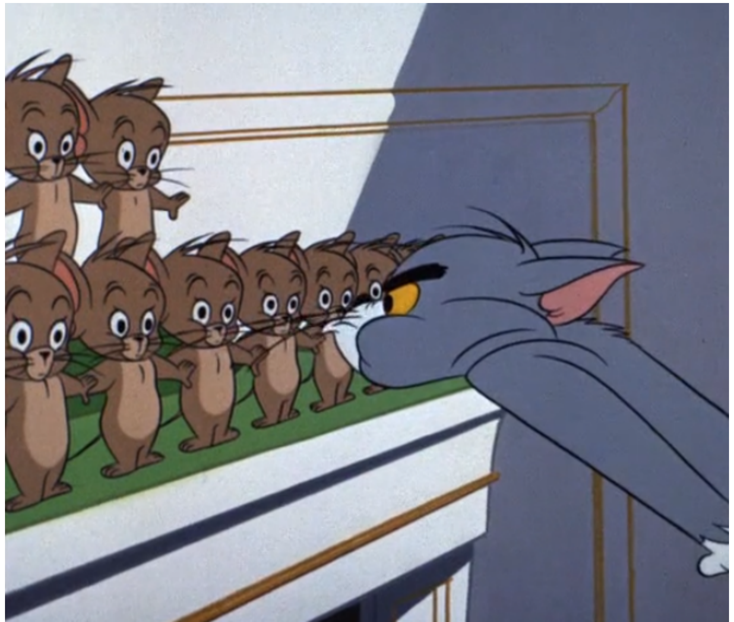
It is the scope（辖域）of the universal quantifier.

$$(\forall x)\boxed{(LIKEJERRY(x))}$$

The Universal Quantifier（全称量词）

DEFINITION

The universal quantifier expresses that a proposition can be satisfied by every member of the domain of discourse, which is interpreted as "given any" or "for all".



x is bound by the universal quantifier.
It is a bound variable（约束变元）.

$$(\forall \boxed{x})(LIKEJERRY(x))$$

The Universal Quantifier（全称量词）

$$(\forall x)(MOUSE(x))$$

The Universal Quantifier（全称量词）

$$(\forall x)(MOUSE(x))$$

FALSE

The Universal Quantifier（全称量词）

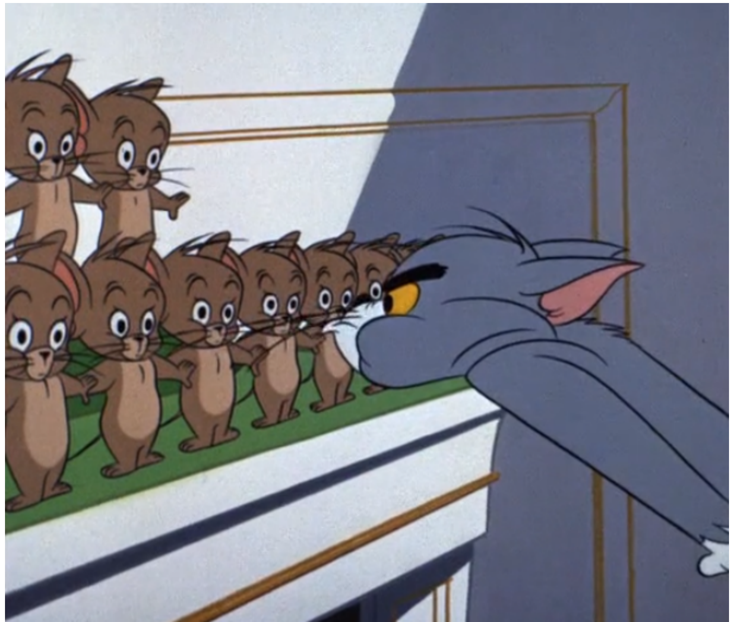
Domain is empty.

$(\forall x)(MOUSE(x))$ TRUE

The Existential Quantifier（存在量词）

DEFINITION

The existential quantifier expresses that a proposition can be satisfied by some member of the domain of discourse, which is interpreted as "there exists", "there is at least one", or "for some".



There exists a real Jerry.

$$(\exists x)(ISJERRY(x))$$

The Existential Quantifier（存在量词）





$$(\exists x)(ELEPHANT(x))$$

The Existential Quantifier（存在量词）





$$(\exists x)(ELEPHANT(x))$$

The Existential Quantifier（存在量词）

Domain is empty.

$$(\exists x)(ELEPHANT(x))$$ FALSE

Quantifier（量词）

The blue rectangle is the scope of "exists x".

$$(\exists x)\boxed{((\forall y)\boxed{(x \leqslant y)})}$$

x and y are all bound variables.

The red rectangle is the scope of "for all y".

Quantifier（量词）

Generally, we don't use repeated names.

The red rectangle is the scope of "exists x".

$$(\exists x)\boxed{P(x)} \vee Q(x)$$

The first x is a bound variable.

The second x is a free variable（自由变元）. It is not bound by any quantifiers.

Quantifier（量词）

Proposition should not include free variables

There are two ways to eliminate free variables

- Replace free variables with constants
- Use quantifiers to convert free variables to bound variables

We can change the name of bound variables（变元易名规则）

$$(\forall y)(LIKEJERRY(y)) = (\forall x)(LIKEJERRY(x))$$

Quantifier（量词）

PROPERTY

1) $(\forall x)(\forall y)P(x,y) = (\forall x)((\forall y)P(x,y))$

2) $(\forall x)(\exists y)P(x,y) = (\forall x)((\exists y)P(x,y))$

3) $(\exists x)(\forall y)P(x,y) = (\exists x)((\forall y)P(x,y))$

4) $(\exists x)(\exists y)P(x,y) = (\exists x)((\exists y)P(x,y))$

5) $(\forall x)(\forall y)P(x,y) = (\forall y)(\forall x)P(x,y)$

6) $(\exists x)(\exists y)P(x,y) = (\exists y)(\exists x)P(x,y)$

Quantifier（量词）

To understand quantifiers better, we can discuss quantifiers with finite domain of discourse.

Quantifier（量词）

Assuming that domain include k elements {1, 2, …, k}.

Then

$$(\forall x)P(x) = P(1) \wedge P(2) \wedge ... \wedge P(k)$$
$$(\exists x)P(x) = P(1) \vee P(2) \vee ... \vee P(k)$$

Exercise

Now the domain is {1, 2}.

$$(\forall x)(\forall y)P(x, y)$$

$$(\exists x)(\exists y)P(x, y)$$

$$(\exists x)(\forall y)P(x, y)$$

$$(\forall y)(\exists x)P(x, y)$$

Quantifier（量词）

$$(\forall x)(\forall y)P(x,y) = P(1,1) \wedge P(1,2) \wedge P(2,1) \wedge P(2,2)$$

$$(\exists x)(\exists y)P(x,y) = P(1,1) \vee P(1,2) \vee P(2,1) \vee P(2,2)$$

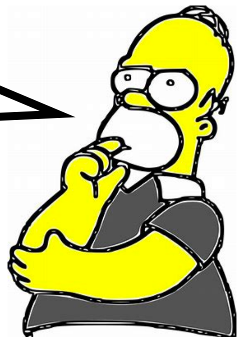
$$(\exists x)(\forall y)P(x,y) = (P(1,1) \wedge P(1,2)) \vee (P(2,1) \wedge P(2,2))$$

$$(\forall y)(\exists x)P(x,y) = (P(1,1) \vee P(2,1)) \wedge (P(1,2) \vee P(2,2))$$

Quantifier（量词）

$$(\exists x)(\forall y)P(x,y) = (P(1,1) \wedge P(1,2)) \vee (P(2,1) \wedge P(2,2))$$
$$(\forall y)(\exists x)P(x,y) = (P(1,1) \vee P(2,1)) \wedge (P(1,2) \vee P(2,2))$$

The two different quantifiers cannot be swapped. But what the relation between them?



Quantifier（量词）

$$
\begin{aligned}
&(\forall y)(\exists x)P(x,y)\\
=&(P(1,1)\vee P(2,1))\wedge(P(1,2)\vee P(2,2))\\
=&((P(1,1)\vee P(2,1))\wedge P(1,2))\vee((P(1,1)\vee P(2,1))\wedge P(2,2))\\
=&(P(1,1)\wedge P(1,2))\vee(P(2,1)\wedge P(1,2))\vee(P(1,1)\wedge P(2,2))\vee(P(2,1)\wedge P(2,2))\\
=&(P(1,1)\wedge P(1,2))\vee(P(2,1)\wedge P(2,2))\vee(P(2,1)\wedge P(1,2))\vee(P(1,1)\wedge P(2,2))\\
=&(\exists x)(\forall y)P(x,y)\vee(P(2,1)\wedge P(1,2))\vee(P(1,1)\wedge P(2,2))
\end{aligned}
$$

$$\Downarrow$$

$$(\exists x)(\forall y)P(x,y)\Rightarrow(\forall y)(\exists x)P(x,y)$$

Quantifier（量词）

$$x1 = 1 \wedge$$
$$x2 = 0 \wedge$$
$$y4 = 1 \wedge$$
$$y5 = 0 \wedge$$
$$x3 = ite(a, x1, x2) \wedge$$
$$y6 = ite(b, y4, y5) \rightarrow$$
$$sum(x3, y6) > 0$$

$$(\forall x1)(\forall x2)(\forall y4)(\forall x3)(\forall y5)(\forall y6)(\forall a)(\forall b)($$
$$x1 = 1 \wedge$$
$$x2 = 0 \wedge$$
$$y4 = 1 \wedge$$
$$y5 = 0 \wedge$$
$$x3 = ite(a, x1, x2) \wedge$$
$$y6 = ite(b, y4, y5) \rightarrow$$
$$sum(x3, y6) > 0)$$

After introducing quantifiers

Thanks & Questions